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(Pages : 4) **N – 1291**

Reg. No. : .....

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Sixth Semester B.Sc. Degree Examination, April 2022

First Degree Programme under CBCSS

Mathematics

Core Course – XI

MM 1643 : ABSTRACT ALGEBRA – RING THEORY

(2018 & 2019 Admission)

Time : 3 Hours Max. Marks : 80

SECTION – A

Answer **all** questions. Each carries **1** mark.

1. Which are the units of Z?
2. Define a zero divisor.
3. Give an example of a non commutative ring.
4. Define an ideal of a ring.
5. State the first isomorphism theorem for rings.
6. Is the ring $2z$ is isomorphic to the ring $3z$?
7. Define the content of a nonzero polynomial.
8. Define an irreducible element in an integral domain.
9. Find the norm of $a + b\sqrt{d}$, where a and b are integers.
10. Define a Noetherian domain.

(10 × 1 = 10 Marks)

P.T.O.

SECTION – B

Answer **any eight** questions. Each carries **2** marks.

11. If a is an element in a ring R, then prove that $a0 = 0a = 0$

12. Prove that if a ring element has a multiplicative inverse, then it is unique.

13. Let a, b and c belong to an integral domain. Prove that if $a \neq 0$ and $ab = ac$, then $b = c$.

14. Show that 0 is the only nilpotent element in an integral domain.

15. Prove that the ideal $\langle x^2 + 1 \rangle$ is maximal in $R[x]$.

16. Prove that the only ideals of a field F are $\{0\}$ and F itself.

17. Determine all ring homomorphisms from z to z.

18. Find a polynomial with integer coefficients that has $\frac{1}{2}$ and $\frac{-1}{3}$ are zeros.

19. Give an example of a field that properly contains the field of complex numbers c.

20. Let F be a field and let a be a non zero element of F. Show that if $f(x+a)$ is irreducible over F, then $f(x)$ is irreducible over F.

21. Construct a field of order 25.

22. Find the kernel of the ring homomorphism φ from $R[x]$ onto c given by $f(x) \to f(i)$.

23. Prove that in an integral domain, every prime is irreducible.

24. Show that $z[x]$ is not a principal ideal domain.

25. Find q and r in $z[i]$ such that $3 - 4i = (2+5i)q + r$ and $d(r) < d(2+5i)$.

26. Show that for any non trivial ideal I of $z[i]$, $\frac{z[i]}{I}$ is finite.

(8 × 2 = 16 Marks)

SECTION – C

Answer **any six** questions. Each carries **4** marks.

27. Let R be a ring. Prove that $a^2 - b^2 = (a+b)(a-b) \forall\ a, b \in R$ if and only if R is commutative.

28. Let R be a ring with unity 1. Show that if 1 has order n under addition, then the characteristic of R is n.

29. Show that a finite commutative ring with no zero divisors and at least two elements has a unity.

30. Find all solutions of the equation $x^3 - 2x^2 - 3 = 0$ in z_{12}.

31. Let φ be a ring homomorphism from a ring R to a ring S. Let A be a subring of R. Prove that if A is an ideal and φ is onto S, then $\varphi(A)$ is an ideal.

32. Let F be a field. If $f(x) \in F[x]$ and $\deg f(x)$ is 2 or 3, then prove that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F.

33. Find the quotient and remainder upon dividing $f(x) = 3x^4 + x^3 + 2x^2 + 1$ by $g(x) = x^2 + 4x + 2$, where $f(x)$ and $g(x)$ belong to $z_5(x)$.

34. Check whether $f(x) = 21x^3 - 3x^2 + 2x + 9$ is an irreducible polynomial over Q.

35. Show that 7 is irreducible in the ring $z[\sqrt{5}]$.

36. Prove that every Euclidean Domain is an Principal Ideal Domain.

37. Prove that in a PID, any strictly increasing chain of ideals $I_2 \subset I_3 \subset \ldots$ must be finite in length.

38. Let D be a Euclidean Domain with measure d. Show that if a and b are associates in D, then $d(a) = d(b)$.

(6 × 4 = 24 Marks)

SECTION – D

Answer **any two** questions. Each carries **15** marks.

39. (a) Let d be an integer. Prove that $z(\sqrt{d}) - \{a + b\sqrt{d} \mid a, b \in z\}$ is an integral domain.

    (b) Define center of a ring R. Prove that the center of a ring is a subring.

40. Let R be a ring and let A be a subring of R. Prove that the set of cosets $\{r + A \mid r \in R\}$ is a ring under the operations $(s + A) + (t + A) = s + t + A$ and $(s + A)(t + A) = st + A$ if and only if A is an ideal of R.

41. State and prove division algorithm for $F[x]$.

42. Let F be a field. Prove that $F[x]$ is a principal ideal domain.

43. Let F be a field and let $p(x) \in F[x]$. Then prove that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F.

44. Show that the ring $Z[\sqrt{-5}]$ is not a Unique Factorisation Domain.

($2 \times 15 = 30$ Marks)
